# MTH 533 Exam 1 Key February 16, 2006

**Short Answer** Give complete answers to 4 of the 5. (3.25 points each)

1. Explain why a convergent sequence $\{\mathbf{x}_k\} \subset \mathbb{R}^n$ is a Cauchy sequence.

   If $\{\mathbf{x}_k\}$ is convergent then for $\epsilon > 0$ there is a positive integer $N$ such that if $k \geq N$ then $\|\,\mathbf{x}_k - \mathbf{a}\,\|_2 < \epsilon/2$. Choose $m, n \geq N$ then

   $$\|\,\mathbf{x}_m - \mathbf{x}_n\,\|_2 \leq \|\,\mathbf{x}_m - \mathbf{a}\,\| + \|\,\mathbf{a} - \mathbf{x}_n\,\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

2. Let $H = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. Suppose $f : H \to \mathbb{R}$ is continuous. Provide full explanations for each of the following.

   (a) $f$ achieves its extreme value on $H$.

   Since $H$ is compact and $f$ is continuous the Extreme Value Theorem implies that $f$ achieves its exteme values on $H$.

   (b) The image $f(H)$ is a closed bounded interval.

   The continuity of $f$ implies $f(H)$ is compact and hence closed and bounded. The set $H$ is connected and so continuity of $f$ implies $f(H)$ is connected. Since the only connected sets of $\mathbb{R}$ are intervals we have that $f(H)$ is a closed bounded interval.

3. Suppose $\displaystyle\lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right) = \lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right) = L$. Does this imply $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$? Prove or find a counterexample.

   No. Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; \quad (x,y) \neq (0,0) \\ 0 & ; \quad (x,y) = (0,0) \end{cases}$ Note that

   $$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0,$$

   but $f(x,x) = \dfrac{1}{2}$.

4. Show that $5x + 3y - 2z = \frac{1}{2}$ is the tangent plane to the function $f(x,y) = \dfrac{2}{\pi}\sin[\pi(x+y)] - xy$ at the point $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

   Note that $f(1/2, -1/2) = 1/4$ and $\nabla f(x,y) = [2\cos[\pi(x+y)] - y, 2\cos[\pi(x+y)] - x]$. Therefore

   $\nabla f(1/2, -1/2) = \left[\dfrac{5}{2}, \dfrac{3}{2}\right]$ and so the normal vector is $\mathbf{n} = [5/2, 3/2, -1]$. The equation of the tangent plane is then given by $\mathbf{n} \cdot [x - 1/2, y + 1/2, z - 1/4] = 0$. Or

   $$\frac{5}{2}\left(x - \frac{1}{2}\right) + \frac{3}{2}\left(y + \frac{1}{2}\right) - z + \frac{1}{4} = 0.$$

Simplifying gives $5x + 3y - 2z = \frac{1}{2}$.

5. Where is the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{f}(x, y) = [2xy, x^2 - y^2]$ one-to-one with a differentiable inverse?

   We have that
   $$D\mathbf{f}(x, y) = \begin{bmatrix} 2y & 2x \\ 2x & -2y \end{bmatrix}$$
   and so the Jacobian $\Delta_{\mathbf{f}}(x, y) = -4(x^2 + y^2) \neq 0$ provided $(x, y) \neq (0, 0)$. Therefore $\mathbf{f}$ will be one-to-one for any $(x, y) \neq (0, 0)$ and in a neighborhood of that point will have a differentiable inverse.

**<u>Problems</u>** Provide complete solutions for <u>6</u> of the 7. (8 points each)

1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. Show if $c \in \mathbb{R}$, then the set $V = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < c\}$ is open.

   Note that the set $B = \{t \in \mathbb{R} : t < c\}$ is open. Indeed, if $t \in B$, set $\delta = (c - t)/2$ then $(t - \delta, t + \delta) \subset B$. Consider the set $f^{-1}(B) = \{x \in \mathbb{R}^n : f(x) = t \text{ for some } t \in B\}$. But, if $f(x) = t$ and $t \in B$ we have $f(x) = t < c$ and so $f^{-1}(B) = \{x \in \mathbb{R}^n : f(x) < c\}$ and since $B$ is open and $f$ is continuous, $f^{-1}(B)$ is open.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
   $$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & ; \quad (x, y) \neq (0, 0) \\ 0 & ; \quad (x, y) = (0, 0) \end{cases}$$

   (a) Show $f(x, y)$ is continuous on all of $\mathbb{R}^2$.

   Since $f$ is a rational function it is continuous for $(x, y) \neq (0, 0)$. We need to show that $f$ is continuous, i.e., $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. Note that
   $$|f(x, y)| \leq \frac{(x^2 + y^2)\sqrt{x^2 + y^2}}{x^2 + y^2} = \sqrt{x^2 + y^2}.$$
   Therefore by the Squeeze Theorem $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

   (b) Let $\mathbf{u} = [u_1, u_2]$ be a unit vector, i.e., $\| \mathbf{u} \|_2 = 1$. Show the directional derivative of $f$ along $\mathbf{u}$ at $(0, 0)$ is
   $$(D_{\mathbf{u}} f)(0, 0) = u_1^2 u_2.$$
   Does this imply that $f$ is differentiable at $(0, 0)$? Prove your answer.

   Note that
   $$\lim_{t\to 0} \frac{f(tu_1, tu_2) - f(0, 0)}{t} = \lim_{t\to 0} \frac{t^3(u_1^2 u_2)}{t\,[t^2(u_1^2 + u_2^2]} = \frac{u_1^2 u_2}{u_1^2 + u_2^2} = u_1^2 u_2$$
   since $u_1^2 + u_2^2 = 1$. This does not imply that $f$ is differentiable at $(0, 0)$. Note that
   $$\lim_{h\to 0} \frac{f(h, 0)}{h} = \lim_{k\to 0} \frac{f(0, k)}{k} = 0$$

and so $f_x(0,0) = f_y(0,0) = 0$ implying that $Df(0,0) = [0,0]$ possibly. For this to hold we must show

$$\lim_{(h,k)\to 0} \frac{f(h,k) - f(0,0) - Df(0,0)\cdot[h,k]}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} = 0.$$

Set $G(h,k) = \dfrac{f(h,k)}{\sqrt{h^2+k^2}} = \dfrac{h^2k}{(h^2+k^2)^{3/2}}$ and note

$$G(h,h) = \frac{h^3}{2^{3/2}h^3} = \frac{1}{2^{3/2}} \neq 0.$$

Therefore $f$ is not differentiable at $(0,0)$.

3. Let $U \subset \mathbb{R}^n$ be a polygonally connected set. A point $\mathbf{a} \in \mathbb{R}^n$ is said to be cluster point of $U$ if and only if for all $\delta > 0$, $B_\delta(\mathbf{a})$ contains infinitely many points of $U$.

   (a) Show that the set of cluster points of $U$ is $\mathring{U} \cup \partial U$.

   Note that $\overline{U} = \mathring{U} \cup \partial U$, and $\overline{U}$ is the set of cluster points of $U$.

   (b) Show $\mathbf{a}$ is a cluster point of $U$ if and only if for $\delta > 0$, $U \cap B_\delta(\mathbf{a})\backslash\{\mathbf{a}\} \neq \emptyset$.

   Let $\mathbf{a}$ be a cluster point of $U$ and let $\delta > 0$. Then since $B_\delta(\mathbf{a}) \cap U$ contains infinitely many points of $U$ so does $B_\delta(\mathbf{a}) \cap U\backslash\{\mathbf{a}\}$, and hence is nonempty. If $B_\delta(\mathbf{a}) \cap U\backslash\{\mathbf{a}\} \neq \emptyset$ for any $\delta > 0$, choose $\mathbf{x}_1 \in B_1(\mathbf{a}) \cap U\backslash\{\mathbf{a}\}$. Set $r_1 = \|\mathbf{a} - \mathbf{x}_1\|_2$ and choose $\mathbf{x}_2 \in B_{r_1}(\mathbf{a}) \cap U\backslash\{\mathbf{a}\}$ then $\mathbf{x}_2 \neq \mathbf{x}_1$. Set $r_2 = \min\{\|\mathbf{a}-\mathbf{x}_1\|_2, \|\mathbf{a}-\mathbf{x}_2\|_2\}$ and choose $\mathbf{x}_3 \in B_{r_2}(\mathbf{a}) \cap U\backslash\{\mathbf{a}\}$. The $\mathbf{x}_3$ is distinct from $\mathbf{x}_2$ and $\mathbf{x}_2$. Continuing in this way gives infinitely many points in $B_\delta(\mathbf{a})$ for all $\delta > 0$.

4. Let $H \subset \mathbb{R}^n$ be nonempty and compact. Suppose $\mathbf{f} : H \to \mathbb{R}^m$ is continuous. Prove $r = \sup\{\|\mathbf{f}(\mathbf{x})\|_2 : \mathbf{x} \in H\}$ is finite and there exists an $\mathbf{x}_0 \in H$ such that $r = \|\mathbf{f}(\mathbf{x}_0)\|_2$.

   Since $\mathbf{f}$ is continuous and $H$ is compact, then $\mathbf{f}(H)$ is compact. This implies that $\mathbf{f}(H)$ is closed and bounded and then $\sup\{\|\mathbf{f}(\mathbf{x})\|_2 : \mathbf{x} \in H\}$ is finite. Also since $g((x)) = \|\mathbf{f}(\mathbf{x})\|_2$ is continuous on $H$, the Extreme Value Theorem implies that $g$ achieves its maximum on $H$ and so there exists a point $\mathbf{x}_0 \in H$ such that $r = g(\mathbf{x}_0)$.

5. Let $f : \mathbb{R}^n \to \mathbb{R}$. $f$ is said to be homogeneous of degree $k$ if $f(t\mathbf{x}) = t^k f(\mathbf{x})$. Show if $f$ is twice continuously differentiable then

$$(\nabla f)(\mathbf{x})\cdot\mathbf{x} = kf(\mathbf{x}) \quad \text{and} \quad \mathbf{x}^{\mathsf{T}}(\nabla^2 f)(\mathbf{x})\mathbf{x} = k(k-1)f(\mathbf{x}).$$

   Here $(\nabla^2 f)(\mathbf{x})$ is the Hessian matrix of second partial derivatives evaluates at $\mathbf{x}$.

   Differentiate $f(t\mathbf{x}) = t^k f(\mathbf{x})$ with respect to $t$, then $\nabla f(t\mathbf{x})\cdot\mathbf{x} = kt^{k-1}f(\mathbf{x})$. Setting $t = 1$ gives $\nabla f(\mathbf{x})\cdot\mathbf{x} = kf(\mathbf{x})$. Note if $u_i = tx_i$ then by the chain rule

$$\frac{\partial f}{\partial u_1}\frac{du_1}{dt} + \cdots + \frac{\partial f}{\partial u_n}\frac{du_n}{dt} = kt^{k-1}f(\mathbf{x}).$$

   Set $t = 1$ gives the same result.

For the second derivative take $n = 2$ and consider $f(tx, ty) = t^k f(x, y)$. Set $u = tx$ and $v = ty$ then $f_u x + f_v y = kt^{k-1} f(x, y)$. Taking second derivatives we have

$$f_{uu}x^2 + f_{uv}xy + f_{vu}xy + f_{vv}y^2 = k(k-1)t^{k-2} f(x, y).$$

Setting $t = 1$ we have

$$f_{xx}x^2 + 2f_{xy}xy + f_{yy}y^2 = k(k-1)f(x, y)$$

which is the same as

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = k(k-1)f(x, y).$$

In general we have

$$f_{u_1u_1}x_1^2 + \cdots + f_{u_nu_n}x_n^2 + 2 \sum_{1 \le i < j \le n} f_{u_iu_j}x_ix_j = k(k-1)t^{k-2} f(\mathbf{x}).$$

Set $t = 1$ we have

$$\mathbf{x}^\mathsf{T} \nabla^2 f(\mathbf{x})\, \mathbf{x} = k(k-1)f(\mathbf{x}).$$

6. Suppose $\mathbf{f}(u, v, w) = [we^u \cos v, we^u \sin v, w^2]$. Explain why $\mathbf{f}$ is one-to-one in an open set containing the point $\mathbf{p} = (0, \pi, 1)$. Find $(D\mathbf{f})(\mathbf{p})$ and $(D\mathbf{f}^{-1})(\mathbf{f}(\mathbf{p}))$.

   Note that

$$D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} we^u \cos v & -we^u \sin v & e^u \cos v \\ we^u \sin v & we^u \cos v & e^u \sin v \\ 0 & 0 & 2w \end{bmatrix}$$

   and so

$$D\mathbf{f}(\mathbf{p}) = \begin{bmatrix} -1 & 0 - 1 & \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Since $\Delta_\mathbf{f}(\mathbf{p}) = 2 \neq 0$, continuity implies that $\Delta_\mathbf{f}(\mathbf{x}) \neq 0$ in an open set containing $\mathbf{p}$. Hence $\mathbf{f}$ is one-to-one in this open set. The Inverse Function Theorem then implies

$$\left(D\mathbf{f}^{-1}\right)(\mathbf{f}(\mathbf{p})) = \begin{bmatrix} -1 & 0 & -1/2 \\ 0 & -1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

7. Given $f(x, y) = (2x + y)e^{-4x^2 - y^2}$, find its local maxima and minima. Are these global extrema? Prove your answer.

   The partial derivatives are

$$f_x = e^{-4x^2-y^2}(2 - 16x^2 - 8xy), \quad f_y = e^{-4x^2-y^2}(1 - 4xy - 2y^2).$$

Solving $f_x = 0,\ f_y = 0$ gives $16x^2 + 8xy = 2,\ 4xy + 2y^2 = 1$. If we multiply the second equation by $-2$ and add it to the first we have $16x^2 - 4y^2 = 0$ and so $y = \pm 2x$. If $y = 2x$ in the first equation we have $32x^2 = 2$ and so $x = \pm 1/4$. If $x = -1/4$ then $y = -1/2$ and if $x = 1/4$, $y = 1/2$. The critical points are then $\left(\frac{-1}{4}, \frac{-1}{2}\right)$ and $\left(\frac{1}{4}, \frac{1}{2}\right)$.

Note if $y = -2x$ in the first equation we have $16x^2 + 8x(-2x) = 2$ which is not possible. The Hessian is given by

$$H(x,y) = e^{-4x^2-y^2} \begin{bmatrix} 64x^2y + 128x^3 - 48x - 8y & 32x^2y + 16xy^2 - 8x - 4y \\ 32x^2y + 16xy^2 - 8x - 4y & 8xy^2 + 4y^3 - 4x - 6y \end{bmatrix}.$$

Now we have that $f_{xx}(-1/4, -1/2) = 12e^{-1/2} > 0$ and $\det H(-1/4, -1/2) = 32e^{-1} > 0$ and so $(-1/4, -1/2, f(-1/4, -1/2)) = \left(-1/4, -1/2, -e^{-1/2}\right)$ is a local minimum. Also $f_{xx}(1/4, 1/2) = -12e^{-1/2} < 0$ and $\det H(1/4, 1/2) = 32e^{-1} > 0$ and so $(1/4, 1/2, f(1/4, 1/2)) = \left(1/4, 1/2, e^{-1/2}\right)$ is a local maximum. Since $\lim_{(x,y)\to\pm(\infty, infty)} f(x,y) = 0$ these values are global extrema. The graph of this function is shown below.