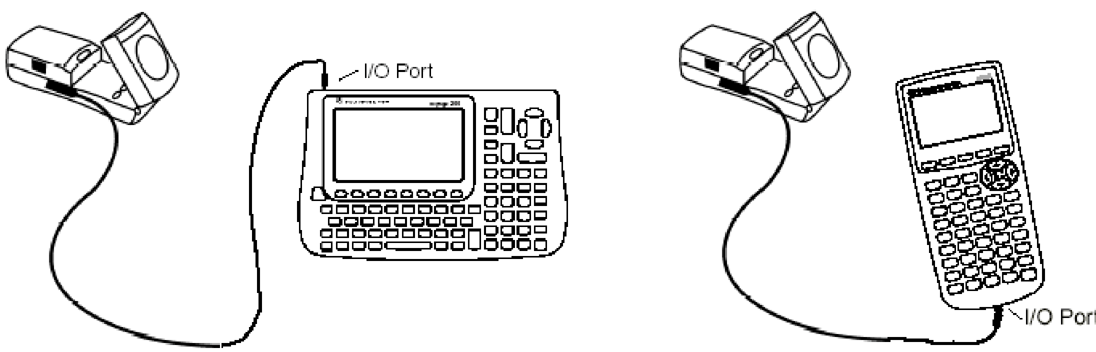
Lab 12: Introduction to Series

The purpose of this laboratory experience is to develop fundamental understanding of series and their convergence or divergence. We will study basic types of series such as geometric and *p*-series and explore the use of a sequence of partial sums as a means to determine convergence.

Pre-Lab

In this lab you will need to use a Calculator-Based Ranger (CBR™) or other device that collects distance data in real-time. On the TI-92+/89 or Voyage 200 we use the Ranger program that can be loaded directly from the CBR™.

To begin, connect the CBR™ to the calculator at the I/O port using the cable provided.

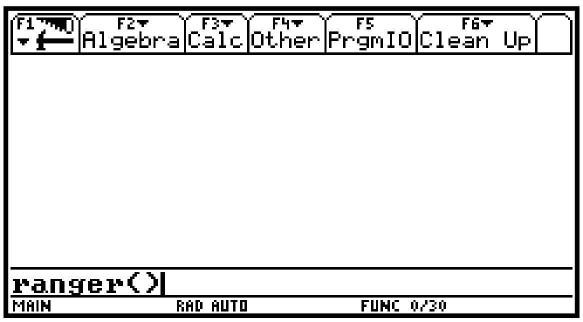


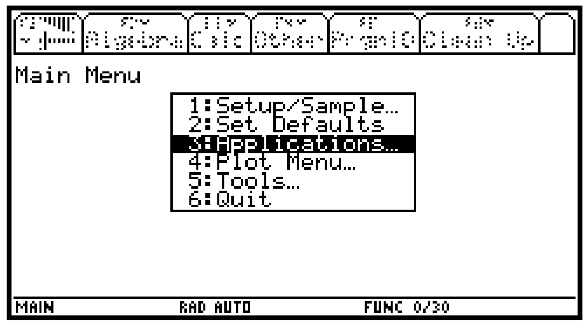
Make sure the calculator is turned on. Open the pivoting head of the CBR and press the (92) key, this will transfer the **Ranger** program into your calculator. From the home screen select the **Ranger** program by pressing [2nd], [VAR-LINK], scrolling to the ranger program and then [ENTER] or simply typing **ranger()** on the command line. Then press [ENTER] to begin the program. The program is menu driven and fairly easy to use. You will be given further instructions on how to use the program in class. Make sure you are familiar enough to connect the CBR™ to your calculator and transfer the program before coming to class the day of the lab.

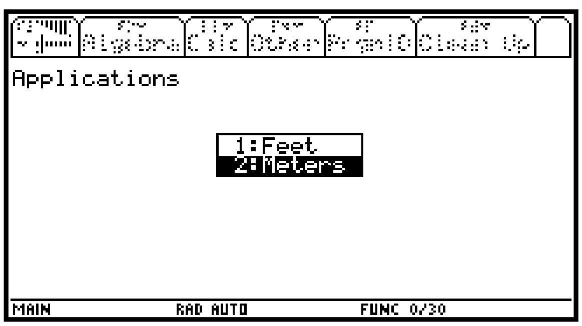
Laboratory Experience

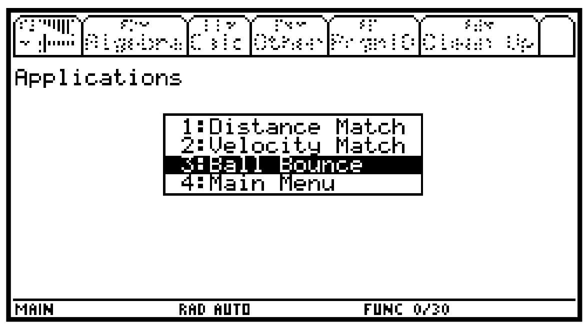
We will begin by investigating the meaning of an infinite series $\sum_{k=1}^{\infty} a_k$ of real numbers a_k. Can we make statements about certain types of series **a priori**? As we did with improper integrals, it makes sense to approach the phenomenon of behavior of series as the number of terms tends toward infinity by investigating *partial sums* as the endpoint of the sum gets larger and larger. Recall that with improper integrals we viewed $\int_1^{\infty} \frac{1}{x^2}dx = \lim_{n\to\infty} \int_1^{n} \frac{1}{x^2}dx$. In a similar manner we can view the sum $\sum_{k=1}^{\infty} a_k$ as $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$. We call this intermediate sum, $S_n = \sum_{k=1}^{n} a_k$, the n^{th} partial sum where the n represents the number of terms in the sum. If the $\lim_{n\to\infty} S_n$ exists, we say that the series *converges* and we define the value of the limit to be the sum of the infinite series. If the limit does not exist, we say the series *diverges*.

1. To begin investigating infinite series, let us consider a phenomenon arising from a bouncing ball. Suppose we drop a ball from a certain height and measure its position over time. One question might come to mind. How much distance does the ball travel before it comes to rest? To begin, we must know something about the rebound height of the ball as it relates to the previous height before the drop. We will use a device called a Calculator-Based Ranger (CBR™) to collect data related to this issue. After you have transferred the **Ranger** program into your calculator, place the command **ranger()** on the command line of the home screen and press [ENTER]. From the **Main Menu** choose **Applications** and then select the units to be meters as shown below.

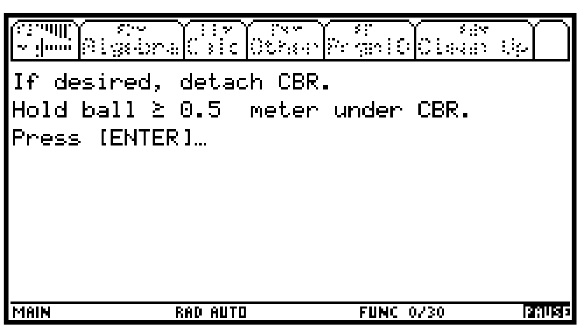


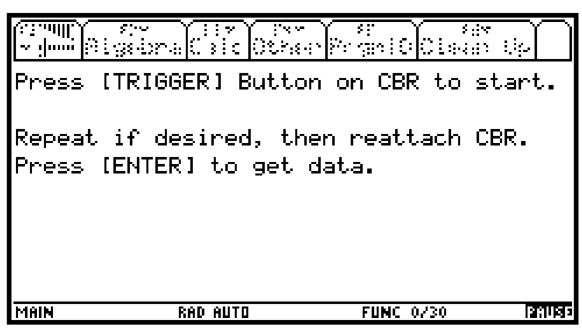






Select the **Ball Bounce** application as shown. You will be given a message telling you that you may detach the CBR™ from the calculator if you wish. You will be using the [TRIGGER] key on the CBR™ to begin data collection.





Now have one of your group members stand on a chair and hold the CBR™ out horizontally with the device pointing down. Have a second group member hold the ball at least 0.5 meters below the CBR™ and release it. Once the ball is released and the releasing person's arm is out of the way, press the TRIGGER key on the CBR™ to begin collecting data. Once the clicking sound from the CBR™ stops, reconnect the CBR™ to the calculator and press ENTER on the calculator. This will transfer data from the CBR™ to your calculator. Your graph should look something like the graph below in Figure 1.

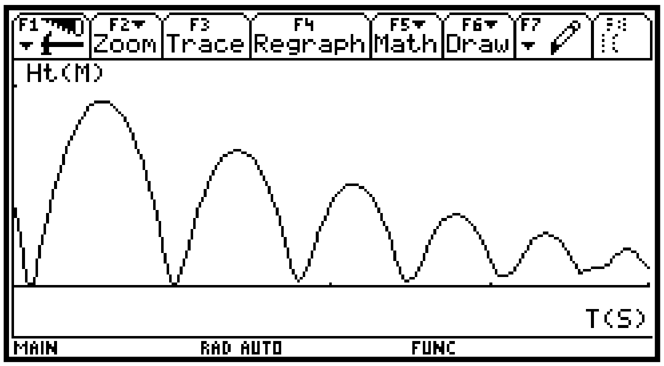


Figure 1: Graph of the Height of a Bouncing Ball

Use the trace feature to find successive peaks on the graph and record your data in the table given. You do not need to worry about the time here since we are only interested in the sequences of heights. Below are two examples where the successive heights were 0.923993 meters and 0.681393 meters respectively.

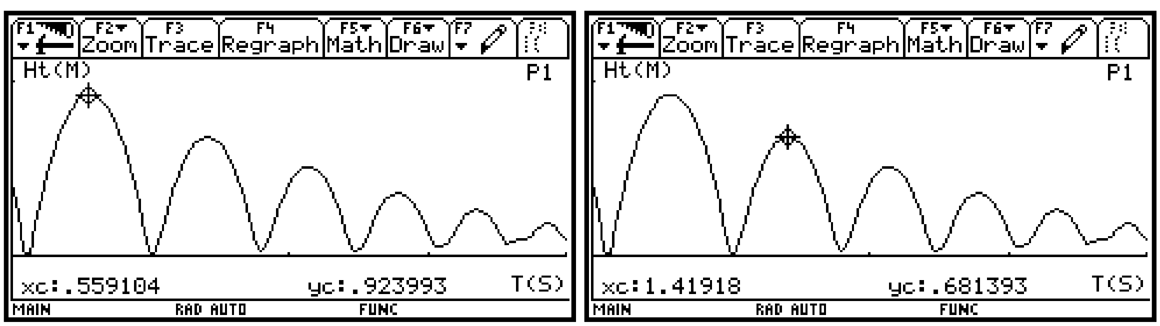


Figure 2: First Two Successive Peaks

Bounce Number	Height of Peak (m)	Ratio (h_{n+1}/h_n)
1		
2		
3		
4		
5		

After you have recorded the peak heights, calculate the ratio of consecutive heights and record the results in the table. For example, for the heights shown above, you would calculate $\frac{0.681393}{0.923993} = 0.737444$ and place this value in the table. The ratios of the heights should be roughly the same and unless you are using Flubber, they should be less than 1. Now average the ratios and record the average value as *r* in the blank provided.

r= ______________________

Suppose we wish to answer the question, what is the total distance a ball will travel if dropped from an initial height h_0 by the time it comes to rest? Notice that for each hump on the graph, the ball must travel the same distance up as it does down except for the initial height h_0 since we are dropping it. For reasons that will be evident later, we will examine the sum as if the ball is going up and down in all cases, even the initial drop. For this reason we will express the initial height as $-h_0 + 2h_0$. Also notice that for each bounce, the ball's height is a fraction of its previous height with that fraction being *r*. We now wish to investigate the sum $-h_0 + 2h_0 + 2rh_0 + 2r^2h_0 + 2r^3h_0 + \cdots$. Discuss this expression in your group until all members understand how the expression is related to the physical situation.

Answering the question of the total distance traveled by the ball now comes down to understanding the sum $-h_0 + 2h_0 \sum_{k=0} r^k$. We will now focus on the infinite sum $\sum_{k=0} r^k$.

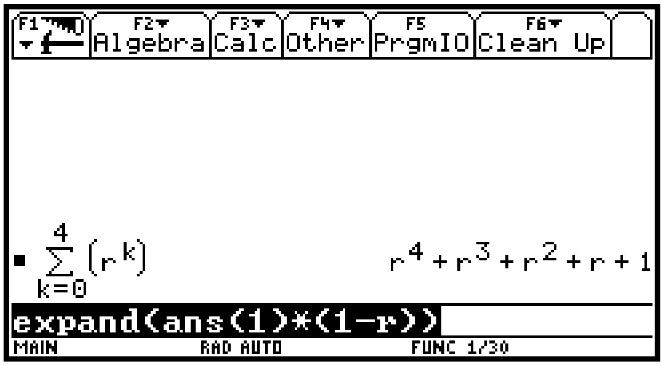
a. Using the value $r = \frac{4}{5}$, investigate the partial sums of the series $\sum_{k=0} r^k$. As you evaluate the partial sums with more and more terms, what appears to be the limiting value? Series of this type are called *geometric series*.

b. Repeat part (a) using $r = \frac{3}{5}$ and $r = \frac{2}{5}$. Discuss any patterns you may see. Find a generalization for the sum $\sum_{k=0} r^k$ where $|r| < 1$.

c. Using your value for *r* found in the ball bounce experiment, find the total distance the ball travels from the time it is dropped to the time it comes to rest.

2. In question 1, you explored a pattern for an infinite sum. Before we can have any confidence in the pattern, we must justify why it would work. Since we are investigating sums of the form $1 + r + r^2 + r^3 + \cdots$, let's look at an algebraic phenomenon we get when we expand the product $(1 + r + r^2 + r^3 + \cdots + r^n)(1 - r)$ for various values of *n*.

   a. First generate the sum $1 + r + r^2 + r^3 + \cdots + r^n$ using the command **∑(r^k,k,0,n)** for a value of n of your own choosing. Below is an example using $n = 4$. Then expand the product of your expression and the binomial $(1 - r)$ using the command **expand(ans(1)*(1–r))**. Record your result.



b. Repeat part (a) for at least 3 other values of *n*, recording your results. Make a generalization for the form of the result.

c. Give an algebraic explanation for why the result will always take on the form conjectured.

3. So far we have investigated the sum $\sum_{k=0} r^k$ where $|r| < 1$. But what if $r = 1$, $r = -1$ or $|r| > 1$?

a. Let $r = 1.01$ and use your computer to study the partial sums. What do you conclude about the convergence or divergence of this series?

b. Now let $r = 1$ and $r = -1$. Explore the convergence or divergence of these series. Discuss your findings.

c. Based on your results from question 2, discuss how your findings from parts (a) and (b) here can be justified.

d. When *r* is negative in a geometric series we get an example of what is called an *alternating series*. Explain why this term is used for series when *r* is negative. Give examples of two alternating geometric series, one convergent and one divergent.

e. Investigate further the behavior of the geometric series for various values of the ratio, *r*. Write a clear and complete statement summarizing your results.

4. We now take a look at series of the form $\sum_{k=1} \frac{1}{k^p}$ where *p* can be any positive real number. Series of this form are called *p*-series.

a. For $p = 0.5$ and $p = 2$, write the first five terms of the series. Now by exploring the partial sums of these series on your CAS, argue that one of the series converges and one diverges. Do not go too wild here on the number of terms in your partial sums since your CAS may take a while to compute the sums. Give an estimate for the sum of the convergent series. Multiply your result by 6 and take the square root. What do you notice? Make a conjecture for this sum.

b. Select two more p values, one of which gives a convergent p-series and the other a divergent one. Give explanations and evidence for your conclusions.

c. When $p = 1$ we get a special case of the p-series. This series is known as the *harmonic series*, $\sum_{k=1} \frac{1}{k}$. Write out and then compute the sum of the first four terms of the harmonic series by hand. Then use your CAS to estimate the partial sums for $n = 4$, $n = 128$, and $n = 1024$.

d. Based on your results from parts (a) and (b), give a preliminary conjecture about the convergence and divergence of the p-series in terms of the positive real number p. Ignore the case of $p = 1$ for now. We will explore it next.

5. We now turn our attention to the *harmonic series*. It is an amazing fact that the harmonic series diverges. You may have been persuaded that it converges by the fact that its terms tend to zero. Any straightforward attempt to compute its sum by a computer, no matter how large or powerful, will lead to the incorrect conclusion that the series converges. In this question, we will use the CAS to obtain a valuable hint about how to prove that the harmonic series diverges.

   a. Building on what you did in 4*c*, make a table of values for the partial sums of the harmonic series with $n = 4, 8, 16, 32, 64$, and 128. Confirm from your table that each time enough successive terms are added for n (the number of terms in the partial sum) to reach the next power of 2, the partial sum increases by an amount exceeding $\frac{1}{2}$.

   b. Assuming that this systematic process of increasing by $\frac{1}{2}$ continues for arbitrarily large n, argue, with no help from the CAS, that the values of the partial sums will eventually exceed 15. Give a value of n for which the partial sum exceeds 15. Justify your answer.

   c. Try to prove what you observed in part (a): that by successively adding enough terms to any partial sum, we can further increase its value by at least $\frac{1}{2}$. Consider

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \text{ and } \frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}.$$

   How many terms are in the first sum and what is the smallest term? What about the second sum? Finally think similarly about the general case,

$$\frac{1}{2^k+1}+\frac{1}{2^k+2}+\cdots+\frac{1}{2^{k+1}}$$

Now put this all together in the clearest and most persuasive argument you can make, and you will have a proof that the harmonic series diverges.

d. Now refine you answer to question 4*d* with a more definitive statement about convergence and divergence of *p*-series.

6. Lastly, we will investigate the *alternating harmonic series*, $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ where we change the signs of all the even numbered terms in the series.

   a. Write out the first four terms in the alternating harmonic series and compute the fourth partial sum by hand.

   b. Use your CAS to approximate the partial sums of this series. Do you think it converges or diverges? Compare your results with $\ln(2)$ and discuss your findings.

We will explore more with the alternating harmonic series later. In particular, we will investigate the curious behavior of the series as we simply rearrange the terms.