Form and Function

Materials: Computer Algebra System, Equilateral Triangle

Goal: This activity explores basic concepts of algebra including properties of algebraic structure. In high school, the primary operations used in algebra were multiplication and addition. Are these the only types of operations that can be used? We will explore this question and look for patterns that are common across algebra.

Recall that in your experiences in elementary school, you first learned to operate on the set of integers, \{…, -3, -2, -1, 0, 1, 2, 3, …\} using a simple operation, addition. When this set was combined with the operation of addition, certain properties held. Some of these properties included: closure, associativity, and an identity.

Consider the set of functions \(T = \{e, f, g, h, j, k\}\) where the functions are defined as follows:

\[
\begin{align*}
    e(x) &= x \\
    f(x) &= \frac{1}{x} \\
    g(x) &= \frac{x}{x-1} \\
    h(x) &= 1 - x \\
    j(x) &= \frac{1}{1-x} \\
    k(x) &= \frac{x-1}{x}
\end{align*}
\]

Mathematics can be described as the study of patterns. For this reason, we often look for similar structure in both nature and mathematical systems. Mathematics can certainly be used to study patterns in the physical world, but we can also look for patterns within mathematics itself. Are there times when one mathematical system is, for all practical purposes, “identical” to another mathematical system and therefore governed by the same properties and relationships? If this is so, then one system can give us quite a bit of information about another system. In fact, if one system is easier to operate on, we can use it instead and then deduce information about the other system without having to do more difficult computations.

In this activity, we will explore what properties exist when pairing the set \(T\) above with the operation of composition of functions, \((T, \circ)\). As you explore this set, try to keep your eyes open for patterns in the relationships among functions. Some properties you may want to check might include: closure, commutativity, associativity, identity, and inverses. For example, consider your experience with basic arithmetic. For addition on the set of integers, the set is closed since adding two integers always results in another integer. The number 0 acts as an identity (i.e. \(a + 0 = 0 + a = a\)) and 5 and -5 are inverses of each other (i.e. \(5 + (-5) = 0\) where 0 is the identity). This type of structure is very important for deducing mathematical truths within a mathematical system.
1. To begin, define the functions above on your computer algebra system. Once the functions are defined, build a table for the composition of functions of the form $A \circ B$ in the space below by simply entering compositions such as $f(g(x)) = \frac{x-1}{x} = k(x)$ and recording the result in the table as shown.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
<th>$j$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$e$</td>
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<td>$f$</td>
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<td>$g$</td>
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<td>$h$</td>
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<td>$j$</td>
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<tr>
<td>$k$</td>
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</tr>
</tbody>
</table>

- a. In general, is the set $T$ closed under the operation of composition of functions? If not, what elements yield an element not in $T$?

- b. In general, do the elements of the set commute with each other? If not, do some of the elements commute with some of the other elements? Explain.

- c. How might you check associativity? How many different permutations of these functions would you need to check to be certain of associativity? Devise a plan to check associativity. Your plan might include other groups from the class (divide and conquer is often very effective).

- d. Is there a function from the set that acts like an identity element? If so, what is the function?
e. Does every element have an inverse? If so, list all elements and their inverses. If
not, list any elements that do have an inverse along with their inverse element.

2. Now consider another set and operation. Take your equilateral triangle and mark the
vertices 1, 2, and 3 on both faces (you will be flipping them over and will need to identify
the same vertex from both faces of the triangle). Orient the triangle as shown in Figure 1.

![Figure 1](image)

a. Keeping the vertices labeled, how many different ways can you orient the triangle
so that one side lies along the horizontal with the opposite vertex pointing upward
(e.g. and are different orientations)?

b. Using the orientations you found in part (a) as the “basic” moves for the triangle,
describe each orientation as a movement such as a flip or rotation from the
original position given in Figure 1. For example, the movement described in part
(a) could be considered a \( \frac{1}{3} \) or 120° counter-clockwise rotation or a \( \frac{2}{3} \) or 240°
clockwise rotation.

c. In order to communicate with each other, we will decide on a common notation
for moves of the triangle. Rotations will be clockwise. We will let \( r_0 \) stand for no
rotation, \( r_1 \) denote a 120° rotation, and \( r_2 \) represent a 240° rotation. For the flips,
we have three axes about which we can flip the triangle (vertical and two
diagonals). We will use \( v, d_1, \) and \( d_2 \) to denote them as shown below.

![Images of triangle orientations]

We will claim that these are the “basic” moves for the triangles so that it comes to
rest back in the same “space” as it started. We will denote this set of moves as
\( M = \{ r_0, v, d_1, r_1, d_2, r_2 \} \). Explain how you know that these are the only “basic”
moves.
d. Build a table for the composition of the moves on the triangle of the form $A \circ B$ where move $B$ is performed first followed by move $A$. In the space below, record the result as shown.

<table>
<thead>
<tr>
<th>Table of $A \circ B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle Movements</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$B$</th>
<th>$A$</th>
<th>$r_0$</th>
<th>$v$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$r_1$</th>
<th>$r_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td></td>
<td></td>
<td>$r_2$</td>
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<tr>
<td>$d_1$</td>
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<tr>
<td>$d_2$</td>
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<tr>
<td>$r_1$</td>
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<tr>
<td>$r_2$</td>
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</tbody>
</table>

e. In general, is the set of moves on the triangle closed under the operation of composition of moves? If not, what elements yield an element not in the original set of moves?

f. In general, do the elements of the set commute with each other? If not, do some of the elements commute with some of the other elements? Explain.

g. How might you check associativity? How many different permutations of these moves would you need to check to be certain of associativity? Devise a plan to check associativity. Your plan might include other groups from the class (divide and conquer is often very effective).

h. Is there a move from the set that acts like an identity element? If so, what is the move?

i. Does every element have an inverse? If so, list all elements and their inverses. If not, list any elements that do have an inverse along with their inverse element.
3. Now consider the two tables you have just created. Examine the answers you gave to the questions about each table (questions 1 a-e and 2 e-i). What similarities do you notice? What differences do you notice?

4. Given your observations about each set and operation, explain how you could use the table for function composition on $T$ to answer questions about set $M$ and its composition of movements on the triangle.

5. Given your observations about each set and operation, explain how you could use the table about set $M$ and its composition of movements on the triangle to answer questions for function composition on $T$.

6. Give a mapping between sets $T$ and $M$ that describes how elements of $T$ have similar behavior as elements in $M$ only with a different operation. What properties of the elements helped to decide what elements from $T$ mapped to elements from $M$?