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Mathematics in Search of History

The history of mathematics is used in a variety of ways to liven up a classroom and give meaning to a lesson. We retell anecdotes, require that students research the lives of mathematicians, encourage students to study the attempts to solve problems of historical interest, and suggest that students discover links between the historical development of science and the development of mathematics. I worry, however, that we are primarily treating the history of mathematics as a fixed entity that consists solely of a set of facts, not as a fluid field within which lively debate occurs and emphases shift over time.

To counter that tendency, I created a mathematical problem in the form of an ancient document. The solution requires that a community of scholars, namely, the students, work together to sort out ambiguities and inconsistencies to arrive at an interpretation of the problem with which the community as a whole is reasonably comfortable. I give this problem to my Advanced Placement calculus class with two major goals in mind:

- To give them an interesting mathematical challenge after the Advanced Placement examination
- To simulate the process by which the history of mathematics is actually developed

I was thrilled by the classroom discussions that took place that year. The remainder of this article presents the problem, as well as the path that the class took as it struggled to make sense of the document. I hope to make the case that this use of the history of mathematics is worthwhile.

THE PROBLEM

In *Archaeology and Language: The Puzzle of Indo-European Origins,* Colin Renfrew (1987) argues that the mother tongue of Indo-European languages originated in agricultural communities that developed in central Anatolia around 8000 B.C.E. He maintains that the spread of language and its differentiation were concomitant with the spread of agriculture, since families, not just agricultural techniques, moved into virgin lands.

As luck would have it, not far from the well-known Neolithic village of Çatal Hüyük in south central Turkey, near the village of Olmazköy, by the banks of Imkânsiz dere, a shepherd recently discovered a cave filled with ancient clay tablets. The writing on the tablets is neither Hurrian, Hattic, Phrygian, Minoan Linear A, nor any other known ancient language. It may be the mother tongue of all Indo-European languages.

Some of the tablets seem to contain mathematical inscriptions. Shown in **figure 1** is a facsimile of what is, perhaps, the most interesting and exciting tablet. It consists of three columns of seven lines of what must be numerical information. Unfortunately, the tablet is broken off at the bottom, and the other part has not yet been recovered. The missing part may still be found in the cave, but it is equally likely that it is by now a bookend in some swank New York City apartment. If this tablet is of the same general size as the rest of the tablets, we have the top two-thirds of it.

Are we treating the history of mathematics as a fluid field?

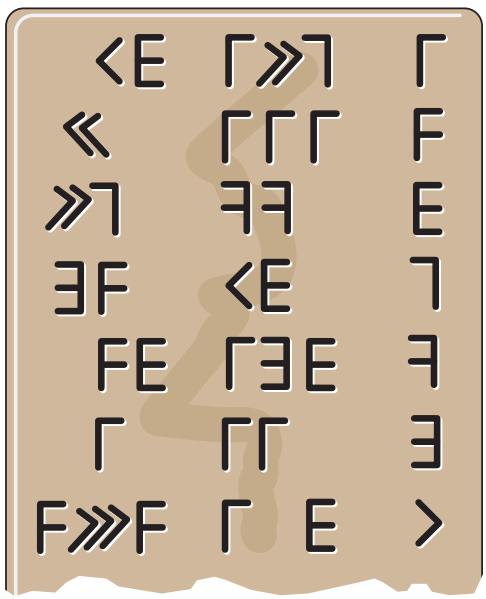

Fig. 1
An ancient tablet

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Your task, should you choose to accept it, is to decipher the tablet. Determine the numbers in the columns and rows, and develop an interpretation of the numerical information. On the basis of your interpretation, determine the contents of the missing third of the tablet.

MY STUDENTS' RESPONSE

Several issues arose simultaneously:

- What is the base?
- Are the numbers written from left to right or from right to left?
- Do the gaps signify zeros or sloppiness?

The "answer man" could only shrug his shoulders and express sympathy with their plight

The entire class assumed that positional notation had been used; no one drew on a knowledge of Roman numerals to suggest otherwise. As for the base, the class decided with little debate that the rightmost column consisted of digits representing the numbers from 1 to 7. Since the first three digits—Γ, F, and E, as well as the next three digits, formed triads generated by adding horizontal segments, it seemed obvious that 7, 8, and 9 were represented by >, ≫, and ⋙, respectively, whereas 10 and 11 were represented by < and ≪, respectively.

But students were not sure whether the numbers were written in base twelve or base thirteen. If all the digits in this system were represented in the tablet, the base was twelve. However, given the grouping of digits in blocks of three, considerable support developed for the idea that 12 would have been represented by ⋘, leading to the conclusion that thirteen was the base. But no one really wanted the base to be thirteen, since unlike base twelve, it was an unfamiliar base that complicated the expression of common fractions.

The students did not know which base to use, so they turned expectantly to me, but the "answer man" was in the delightful position of being able to shrug his shoulders and express sympathy with their plight, but nothing more. I did point out that if they were a community of archaeologists studying a real tablet, either a consensus would emerge after they had explored various hypotheses, or it would not. I also reminded them of the principle known as *Occam's razor,* which states that the simplest theory that fits the data is the one that ought to be chosen.

The students quickly shook off their intellectual paralysis and chose twelve as the base, not as an answer but as a point of departure, realizing that if twelve did not lead to fruitful results, they would have to abandon it. Similarly, they had no way of knowing whether the numbers were written from left to right or from right to left, so they had to explore both possibilities.

Figures 2 and **3** show their two-step translation into base ten. As shown in **figure 2,** the students first wrote the digits in our Hindu-Arabic notation. They next converted the numbers to base ten. If the numbers were written from left to right, the numbers in the first line, for example, would be 10 • 12 + 3 = 123 and 1 • 144 + 8 • 12 + 4 = 244. Since they could not agree about whether an interior space meant a zero, they gave two values for the middle number in line 7. The class obtained the results shown in **figure 3**.

0	(10)	3	1	8	4	1
0	(11)		1	1	1	2
8	4		5	5		3
6	2		(10)	3		4
0	2	3	1	6	3	5
0	1		1	1		6
2	9	2	1		3	7

Fig. 2
The students converted the numerals into a place-value system.

123	244	1
132 or 11	157	2
1200 or 100	780 or 65	3
888 or 74	1476 or 123	4
27	219	5
12 or 1	156 or 13	6
398	147 or 15	7

Fig. 3
The base-twelve numerals in figure 2 converted into our base-ten system

An initial burst of excitement occurred when someone realized that 1200/780 = 100/65 and 888/1476 = 74/123, but it vanished when we realized that the result arose from nothing more than the cancellation of bases. Then a student noticed that the difference between some entries in the first and second columns were perfect squares, that is, 244 – 123 = 121, 157 – 132 = 25, 123 – 74 = 49, and 156 – 12 = 144. But, unfortunately, 219 – 27 = 192, not 196, and the entries in rows 3 and 7 came nowhere near such a result.

However, the notion that some differences were perfect squares lent great support to the belief that 132 was the correct reading for line 2, that 74 and 123 were the correct readings for line 4, and that 12 and 156 were the correct readings for line 6. On that basis, lines 3, 5, and 7 contained mistakes. One student then concluded that the tablet was obviously the work of a student, not a teacher, so for a brief period we discussed, in general terms, the authorship and purpose of the tablet. The students were disheartened by the likelihood that lines 3, 5, and 7 contained mistakes, but I did point out that viable explanations of mistakes can be extremely useful in establishing the validity of an interpretation of an ancient document. This idea was a new one to them but one that they quickly valued in this context.

Then a truly provocative question emerged—in looking for perfect squares, are we imposing a pattern on the tablet that is not, in fact, there? In other words, are we labeling as mistakes those entries that are correct if the tablet is properly interpreted? We never pursued these questions in detail because the class turned impatiently to an exploration of a right-to-left reading, leaving me with lots to say and no audience.

Writing the digits from right to left gave the results shown in **figure 4.** Converting to base ten gave **figure 5.**

3	(10)	0	4	8	1	1
0	(11)	0	1	1	1	2
	4	8		5	5	3
	2	6		3	(10)	4
3	2	0	3	6	1	5
	1	0		1	1	6
2	9	2	3		1	7

Fig. 4
The numbers in figure 2, using a right-to-left numeration system

552 or 46	673	1
132 or 11	157	2
56	65	3
30	46	4
456 or 38	505	5
12 or 1	13	6
398	433 or 37	7

Fig. 5
The numbers in figure 4 converted into base ten

Suddenly, insights started popping up all over the room. We were already primed to look at differences, so students quickly saw the following:

$$673 - 552 = 121$$
$$157 - 132 = 25$$
$$65 - 56 = 9$$
$$46 - 30 = 16$$
$$505 - 456 = 49$$
$$13 - 12 = 1$$

Given the great number of differences yielding perfect squares, the class quickly abandoned the left-to-right hypothesis and eliminated any alternative readings in columns 1 and 2 that did not yield a perfect-square difference. Thus, because “there’s more stuff going on with the right-to-left hypothesis,” they chose to work with the numbers shown in **figure 6.**

552	673	1
132	157	2
56	65	3
30	46	4
456	505	5
12	13	6
398	433	7

Fig. 6
The numbers that the students finally chose for the tablet

Only line 7 was then problematic, since 433 – 398 = 35, not 36. Was 433 a mistake, was 398 a mistake, or were they both in error? One student argued that the writer meant to write F E, which is 434, not Γ E, which is 433, because 434 – 398 = 36. This argument quickly gained adherents because it explained the error as a simple, careless mistake, dropping the middle bar, but then someone noticed that all elements in the left-hand column were even, whereas all but the number 46 in the middle column were odd. That result suggested that the error might be with F ⋙ F, not with Γ E.

One student said that if the author of the tablet had written ≫ F = 2 • 144 + 8 • 12 = 384 instead of F ⋙ F = 2 • 144 + 9 • 12 + 2 = 398, then the difference, 433 – 384 = 49, would have been a perfect square. Another noted that if the author had written < F = 2 • 144 + 10 • 12 = 408, then the difference, 433 – 408 = 25, would also have been a perfect square. But these errors were more difficult to explain than erroneously writing Γ in place of F; so at that point, the class was very uncertain about the proper entries for line 7. The fact that 46 was the only even number in the second column bothered them, so they turned to line 4 and quickly discovered that if the author had written < F = 2 • 12 + 10 = 34 instead of < E = 3 • 12 + 10 = 46, we could also obtain a perfect square as a difference, namely, 34 – 30 = 4. But since 4 is not odd, this result was not consistent with other patterns.

We never did get around to considering how 46 could be turned into an odd number n such that $n - 30$ gives a perfect square, because John Maglio,

Were we imposing a pattern on the tablet that was, in fact, not there?

a quiet but thoughtful observer of our discussions, suddenly dispelled all doubts with two brilliant insights. First he noted that if we adopted 34 for line 4 and 408 for line 7, the sum of the first two entries in each line, in addition to the difference, was a perfect square in all cases:

$$
\begin{aligned}
673 + 552 &= 1225 \\
157 + 132 &= 289 \\
65 + 56 &= 121 \\
34 + 30 &= 64 \\
505 + 456 &= 961 \\
13 + 12 &= 25 \\
433 + 408 &= 841
\end{aligned}
$$

The path taken by the class resembled the discovery of the meaning of Plimpton 322

The class grew deliciously quiet for a second, and then everyone started talking at once. They really liked this idea and concluded that it was important enough that it should supersede the odd-even pattern. But John gained even greater support for his reading of the tablet when he explained that with 34 and 408, the entries in the tablet formed legs and hypotenuses of right triangles. In line 6, we have the entries for a 5-12-13 right triangle. By using 34 in line 4, we have an 8-15-17 right triangle. **Figure 7** shows the Pythagorean triples that John found.

385	552	673	1
85	132	157	2
33	56	65	3
16	30	34	4
217	456	505	5
5	12	13	6
145	408	433	7

Fig. 7
John's Pythagorean triples

John's discoveries came on the last day of class. This community of scholars—believing that it had come up with a viable, corrected set of entries for the table—ended its analysis on a high note. My students thought that they had arrived at a plausible explanation of why 46 appeared instead of 34, and they were convinced that not all entries in the next-to-last column had to be odd. They were not comfortable with their explanation of how 398 came to be written instead of 408, but they strongly believed that 408 and 433 were correct. Unfortunately, we did not have time to study the tablet further and could not consider whether the tablet was a random or ordered collection of Pythagorean triples. Nor did we have time to determine the entries in the missing bottom third of the tablet. We leave that crucial task to the reader and to other classroom communities of scholars.

In thinking about the path taken by this class, I was struck by how closely it resembled the discovery of the meaning of Plimpton 322, a Babylonian tablet dating to around 1800 B.C.E. Unfortunately, Plimpton 322 is usually presented as a sequential arrangement of fifteen Pythagorean triples, not as a document whose difficulties and ambiguities led scholars through a maze of conjectures until the differing interpretations of Neugebauer (1957) and Bruins (1949, 1957) were reached. In developing their interpretations, Neugebauer and Bruins used mathematics to construct the history of Babylonian mathematics.

Similarly, in using their mathematical knowledge to create an interpretation of a tablet, my students are writing or constructing history instead of just talking about it. I hope that the three days that we spent on this tablet revealed to them the kinds of thinking, swashbuckling explorations, and community effort that are involved in developing the history of mathematics.

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